Directional Technology Distance Functions: Theory and Applications

by

Lane Blume Hudgins and Daniel Primont

Department of Economics, MailCode 4515
Southern Illinois University Carbondale
Carbondale, IL 62901-4515

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1. Introduction

Properties of the directional technology distance function have been given in a paper by Chambers, Chung, and Färe (1998). This function, $\vec{D}(x, y; g_x, g_y)$, is an implicit representation of an M-output, N-input production technology. An input-output vector, (x, y), is feasible if and only if $\vec{D}(x, y; g_x, g_y) \geqq 0$, where (g_x, g_y) is a "direction" vector to be described later. An important antecedent of the directional technology distance function is the shortage function, introduced by Luenberger (1992, 1995).

In this paper the theory of the directional technology distance function is extended by deriving a set of restrictions on the first and second derivatives of the directional technology distance functions. These restrictions would be useful in building an econometric model based on the directional technology distance function. In a second application, it is shown that the usual comparative static results for a competitive firm are easily established.

Let $x \in R_+^N$ be the input vector and let $y \in R_+^M$ be the output vector. The technology T is given by

$$T = \{(x, y) : \ x \text{ can produce } y\}.$$

Assume (see Chambers, Chung, Färe (1998))

- T1. T is closed
- T2. Free disposability: if $(x, y) \in T, x' \geq x$, and $y' \leq y$ then $(x', y') \in T$.
- T3. No free lunch: if $(x, y) \in T$ and $x = 0$ then $y = 0$.
- T4. Possibility of inaction: $(0, 0) \in T$.
- T5. T is convex.

The directional technology distance function is a particular representation of a multi-output, multi-input production technology. Following Chambers, Chung, and Färe (1998),

$$\vec{D}(x,y;g_x,g_y) = \begin{cases} \max\{\beta : (x,y)+\beta(-g_x,g_y)\in T\} \\ \qquad \text{if } (x,y)+\beta(-g_x,g_y)\in T \text{ for some } \beta \\ \\ -\infty \quad \text{otherwise.} \end{cases} \tag{1}$$

The calculation of the directional technology distance function is depicted in Figure 1.

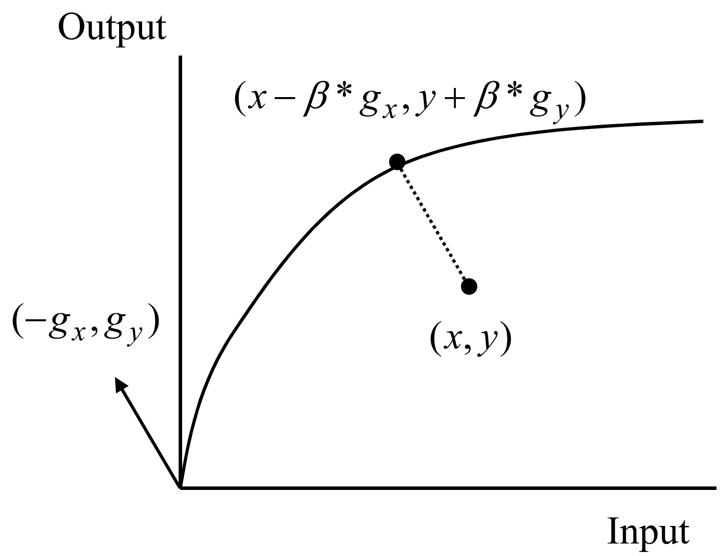

Figure 1

where $\beta^* = \vec{D}(x,y;g_x,g_y)$.

There are, of course, many different implicit representations of a multi-output, muliti-input production technology. However, the directional technology distance function is particularly well-suited to the task of providing a measure of technical efficiency in the full input-output space. To see this consider some of the competing alternative measures.

The hyperbolic measure, proposed by Färe, Grosskopf, and Lovell (1985), is given by

$$F_g(x,y) = \min\left\{\lambda : \left(\lambda x, \frac{y}{\lambda}\right)\in T\right\}.$$

The calculation of this hyperbolic measure is depicted in Figure 2.

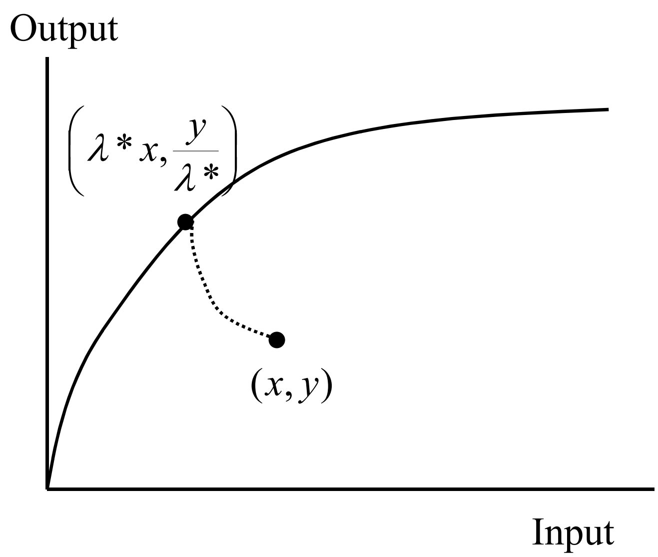

Figure 2

where $\lambda^* = F_g(x, y)$. It is possible to give this measure an economic interpretation but this is done at the expense of assuming constant returns to scale. For the details see Färe, Grosskopf, and Zaim (2002).

Another possibility is the radial measure given by

$$F_R(x, y) = \max\left\{\delta : (\delta x, \delta y) \in T\right\}.$$

The calculation of this measure is depicted in Figure 3.

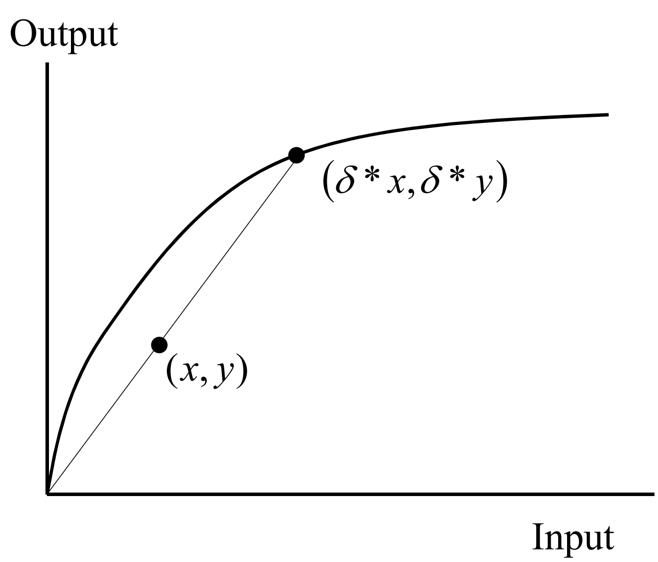

Figure 3

However, this measure could produce very large values (high inefficiency scores) even when (x, y) is very close to the frontier. Moreover, this measure completely breaks down under constant returns to scale.

Lemma 2.2 in Chambers, Chung, and Färe (1998) establishes that A1 - A5 imply the following properties:

D1. Translation Property

$$\vec{D}(x - \alpha g_x, y + \alpha g_y; g_x, g_y) = \vec{D}(x, y; g_x, g_y) - \alpha \text{ for all } \alpha \in R$$

D2. g-Homogeneity of Degree Minus One

$$\vec{D}(x, y; \lambda g_x, \lambda g_y) = \lambda^{-1}\vec{D}(x, y; g_x, g_y), \lambda > 0$$

D3. Input Monotonicity

$$x' \geq x \Rightarrow \vec{D}(x', y; g_x, g_y) \geq \vec{D}(x, y; g_x, g_y)$$

D4. Output Monotonicity

$$y' \geq y \Rightarrow \vec{D}(x, y'; g_x, g_y) \leq \vec{D}(x, y; g_x, g_y)$$

D5. Concavity

$$\vec{D}(x, y; g_x, g_y) \text{ is concave in } (x, y)$$

2. Derivative Properties and Econometric Modelling

An econometric model of the directional technology distance function should impose properties D1 - D5 listed above. This is conveniently accomplished by imposing the restrictions on the first and second derivatives of $\vec{D}(x, y; g_x, g_y)$ that are implied by D1 - D5. These derivative conditions are given in the following lemma.

Lemma 1: Assume that $\vec{D}(x, y; g_x, g_y)$ is twice continuously differentiable. Then D1 - D5 imply that:

DD1. Translation Property

$$\nabla_x \vec{D}(x, y; g_x, g_y) g_x - \nabla_y \vec{D}(x, y; g_x, g_y) g_y = 1$$

DD2. g-Homogeneity of Degree Minus One

$$\nabla_{g_x} \vec{D}(x, y; g_x, g_y) + \nabla_{g_y} \vec{D}(x, y; g_x, g_y) = -\vec{D}(x, y; g_x, g_y)$$

DD3. Input Monotonicity

$$\nabla_x \vec{D}(x, y; g_x, g_y) \geq 0$$

DD4. Output Monotonicity

$$\nabla_y \vec{D}(x, y; g_x, g_y) \leq 0$$

DD5. Concavity

$$H_{\vec{D}} \text{ is negative semidefinite}$$

DD6. Symmetry

$$H_{\vec{D}} \text{ is symmetric}$$

where

$$H_{\vec{D}} = \left[\begin{array}{cc} \nabla_{xx}\vec{D}(x, y; g_x, g_y) & \nabla_{xy}\vec{D}(x, y; g_x, g_y) \\ \nabla_{yx}\vec{D}(x, y; g_x, g_y) & \nabla_{yy}\vec{D}(x, y; g_x, g_y) \end{array} \right]$$

is the Hessian matrix of $\vec{D}$.

Proof: Differentiating (D1) with respect to α we get

$$-\nabla_x \vec{D}(x - \alpha g_x, y + \alpha g_y; g_x, g_y)g_x + \nabla_y \vec{D}(x - \alpha g_x, y + \alpha g_y; g_x, g_y)g_y = -1.$$

Set α equal to one and multiply by -1 to get DD1:

$$\nabla_x \vec{D}(x, y; g_x, g_y)g_x - \nabla_y \vec{D}(x, y; g_x, g_y)g_y = 1.$$

D2. says that the directional technology distance function is homogeneous of degree minus one in (g_x, g_y). DD2 follows by Euler's Theorem. DD3 and DD4 follow directly from the monotonicity conditions, D3 and D4, respectively. DD5 follows directly from the concavity of $\vec{D}(x, y; g_x, g_y)$ in (x, y) and DD6. follows from Young's Theorem. QED

Before concluding this section there is one more interesting property to explore. The profit function is defined as

$$\Pi(p, w) = \max_{x,y} \{py - wx : (x, y) \in T\} \tag{2}$$

$$= \max_{x,y} \left\{py - wx : \vec{D}(x, y; g_x, g_y) \geq 0\right\} \tag{3}$$

since

$$(x, y) \in T \Leftrightarrow \vec{D}(x, y; g_x, g_y) \geq 0. \tag{4}$$

Because of (1) and (4) we can write

$$(x, y) \in T \Leftrightarrow (x - \vec{D}(x, y; g_x, g_y)g_x, y + \vec{D}(x, y; g_x, g_y)g_y) \in T,$$

by the free disposability assumption. Thus, profit may be defined by the unconstrained maximization problem:

$$\begin{aligned} \Pi(p, w) &= \max_{x,y} \left\{p\left(y + \vec{D}(x, y; g_x, g_y)g_y\right) - w\left(x - \vec{D}(x, y; g_x, g_y)g_x\right)\right\} \\ &= \max_{x,y} \left\{py - wx + \vec{D}(x, y; g_x, g_y)\left(pg_y + wg_x\right)\right\} \end{aligned}$$

The first order conditions are:

$$\begin{aligned} -w + \nabla_x \vec{D}(x, y; g_x, g_y)\,(pg_y + wg_x) &= 0 \\ p + \nabla_y \vec{D}(x, y; g_x, g_y)\,(pg_y + wg_x) &= 0 \end{aligned}$$

or

$$\frac{w}{pg_y + wg_x} = \nabla_x \vec{D}(x, y; g_x, g_y) \tag{5}$$

$$\frac{p}{pg_y + wg_x} = -\nabla_y \vec{D}(x, y; g_x, g_y) \tag{6}$$

These are the inverse supply and demand functions. Prices (w, p) are normalized by the number, $pg_y + wg_x$.

While this approach is efficient it does not provide an economic interpretation of the term, $pg_y + wg_x$. To provide such an interpretation we turn to a more traditional treatment of the profit maximization problem. Write the Lagrangian function for (3) as

$$L = py - wx + \lambda \vec{D}(x, y; g_x, g_y)$$

First order conditions are:

$$\begin{aligned} L_x &= -w + \lambda \nabla_x \vec{D}(x, y; g_x, g_y) = 0 \Rightarrow \nabla_x \vec{D}(x, y; g_x, g_y) = \frac{w}{\lambda} > 0 \\ L_y &= p + \lambda \nabla_y \vec{D}(x, y; g_x, g_y) = 0 \Rightarrow \nabla_y \vec{D}(x, y; g_x, g_y) = \frac{-p}{\lambda} < 0 \end{aligned} \tag{7}$$

or

$$wg_x = \lambda \nabla_x \vec{D}(x, y; g_x, g_y) g_x \tag{8}$$

$$pg_y = -\lambda \nabla_y \vec{D}(x, y; g_x, g_y) g_y \tag{9}$$

Multiplying (DD1) by λ we get:

$$\lambda \nabla_x \vec{D}(x, y; g_x, g_y) g_x - \lambda \nabla_y \vec{D}(x, y; g_x, g_y) g_y = \lambda$$

thus, adding (8) and (9) we get:

$$\begin{aligned} pg_y + wg_x &= \lambda \nabla_x \vec{D}(x, y; g_x, g_y) g_x - \lambda \nabla_y \vec{D}(x, y; g_x, g_y) g_y \\ &= \lambda \\ &\Rightarrow \lambda = pg_y + wg_x \end{aligned} \tag{10}$$

Thus, $pg_y + wg_x$ is the optimal value of the Lagrangian multiplier in the profit maximization problem. If the technology is perturbed (improved) by a small value, ε, from

$$T = \left\{ (x, y) : \vec{D}(x, y; g_x, g_y) \geq 0 \right\}$$

to

$$T' = \left\{ (x,y) : \vec{D}(x,y;g_x,g_y) + \varepsilon \geq 0 \right\}$$

then the firm's profit will rise and $\frac{\partial \Pi(p,w)}{\partial \varepsilon} = pg_y + wg_x$.[1]

Putting (10) into (7) and rearranging we get

$$\begin{aligned} \frac{w}{pg_y + wg_x} &= \nabla_x \vec{D}(x,y;g_x,g_y) \\ \frac{p}{pg_y + wg_x} &= -\nabla_y \vec{D}(x,y;g_x,g_y) \end{aligned}$$

which, of course, is the same result as (5) and (6).

3. Comparative Statics

In this section we show how comparative static derivatives of the input demand and the output supply functions may be expressed as functions of the first and second order derivatives of the directional technology distance function. Rearranging (5) and (6), we get

$$\nabla_x \vec{D}(x,y)\,(pg_y + wg_x) = w \tag{11}$$

$$\nabla_y \vec{D}(x,y)\,(pg_y + wg_x) = -p \tag{12}$$

First, differentiate (11) and (12) with respect to the input price vector, w.

$$\begin{aligned} \nabla_x \vec{D}(x,y)g_x + \left[\nabla_{xx}\vec{D}(x,y)\frac{\partial x}{\partial w} + \nabla_{xy}\vec{D}(x,y)\frac{\partial y}{\partial w} \right](pg_y + wg_x) &= 1 \\ \nabla_y \vec{D}(x,y)g_x + \left[\nabla_{yx}\vec{D}(x,y)\frac{\partial x}{\partial w} + \nabla_{yy}\vec{D}(x,y)\frac{\partial y}{\partial w} \right](pg_y + wg_x) &= 0 \end{aligned}$$

and write the result, rearranged, n matrix notation,

$$\begin{bmatrix} \nabla_{xx}\vec{D}(x,y) & \nabla_{xy}\vec{D}(x,y) \\ \nabla_{yx}\vec{D}(x,y) & \nabla_{yy}\vec{D}(x,y) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial w} \end{bmatrix} = \frac{1}{pg_y + wg_x} \begin{bmatrix} 1 - \nabla_x \vec{D}(x,y)g_x \\ -\nabla_y \vec{D}(x,y)g_x \end{bmatrix}.$$

Next, differentiate (11) and (12) with respect to output prices, p.

$$\begin{aligned} \nabla_x \vec{D}(x,y)g_y + \left[\nabla_{xx}\vec{D}(x,y)\frac{\partial x}{\partial p} + \nabla_{xy}\vec{D}(x,y)\frac{\partial y}{\partial p} \right](pg_y + wg_x) &= 0 \\ \nabla_y \vec{D}(x,y)g_y + \left[\nabla_{yx}\vec{D}(x,y)\frac{\partial x}{\partial p} + \nabla_{yy}\vec{D}(x,y)\frac{\partial y}{\partial p} \right](pg_y + wg_x) &= -1. \end{aligned}$$

[1] It is also possible to infer this result from the proof in the Appendix of Chambers, Chung, and Färe (1998).

Rearrange and write in matrix notation.

$$\begin{bmatrix} \nabla_{xx}\vec{D}(x,y) & \nabla_{xy}\vec{D}(x,y) \\ \nabla_{yx}\vec{D}(x,y) & \nabla_{yy}\vec{D}(x,y) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial p} \\ \\ \dfrac{\partial y}{\partial p} \end{bmatrix} = \frac{1}{pg_y + wg_x} \begin{bmatrix} -\nabla_x\vec{D}(x,y)g_y \\ -1-\nabla_y\vec{D}(x,y)g_y \end{bmatrix}.$$

$$\begin{bmatrix} \nabla_{xx}\vec{D}(x,y) & \nabla_{xy}\vec{D}(x,y) \\ \nabla_{yx}\vec{D}(x,y) & \nabla_{yy}\vec{D}(x,y) \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial w} & \dfrac{\partial x}{\partial p} \\ \\ \dfrac{\partial y}{\partial w} & \dfrac{\partial y}{\partial p} \end{bmatrix}$$

$$\begin{aligned}
&= \frac{1}{pg_y + wg_x} \begin{bmatrix} 1-\nabla_x\vec{D}(x,y)g_x & -\nabla_x\vec{D}(x,y)g_y \\ -\nabla_y\vec{D}(x,y)g_x & -1-\nabla_y\vec{D}(x,y)g_y \end{bmatrix} \\
&= \frac{1}{pg_y + wg_x} \begin{bmatrix} -\nabla_y\vec{D}(x,y)g_y & -\nabla_x\vec{D}(x,y)g_y \\ -\nabla_y\vec{D}(x,y)g_x & -\nabla_x\vec{D}(x,y)g_x \end{bmatrix} \qquad \text{(using DD1.)} \\
&= \frac{-1}{pg_y + wg_x} \begin{bmatrix} \nabla_y\vec{D}(x,y)g_y & \nabla_x\vec{D}(x,y)g_y \\ \nabla_y\vec{D}(x,y)g_x & \nabla_x\vec{D}(x,y)g_x \end{bmatrix}
\end{aligned}$$

Thus, the matrix of comparative static derivatives of the input demand and the output supply functions can be found above after we invert the Hessian matrix of the directional technology distance function. We get,

$$\begin{bmatrix} -\nabla_{ww}\Pi(p,w) & -\nabla_{wp}\Pi(p,w) \\ \\ \nabla_{pw}\Pi(p,w) & \nabla_{pp}\Pi(p,w) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial w} & \dfrac{\partial x}{\partial p} \\ \dfrac{\partial y}{\partial w} & \dfrac{\partial y}{\partial p} \end{bmatrix}$$

$$= \frac{-1}{pg_y + wg_x} \begin{bmatrix} \nabla_{xx}\vec{D}(x,y) & \nabla_{xy}\vec{D}(x,y) \\ \nabla_{yx}\vec{D}(x,y) & \nabla_{yy}\vec{D}(x,y) \end{bmatrix}^{-1} \begin{bmatrix} \nabla_y\vec{D}(x,y)g_y & \nabla_x\vec{D}(x,y)g_y \\ \nabla_y\vec{D}(x,y)g_x & \nabla_x\vec{D}(x,y)g_x \end{bmatrix}$$

4. Closing Remarks

In this paper we have established the derivative restrictions on the directional technology distance function that would be useful in econometric work. It was also shown that the standard neoclassical comparative static analysis for a competitive firm can be easily handled with the directional technology distance function. Other applications are possible. For example, Färe and Grosskopf (2000) show, among other things, that the directional technology distance function can be used to model plant capacity. For another example, Färe and Primont (2003) use the directional technology distance function to find conditions under which productivity indicators for each firm in an industry can be aggregated to a productivity indicator for the industry as a whole.

References

[1] Chambers, Robert, Yangho Chung, and Rolf Färe , "Profit, Directional Distance Functions, and Nerlovian Efficiency," *Journal of Optimization Theory and Applications*, Vol. 98, No. 2 (August 1998), pp. 351-364.

[2] Färe, Rolf and Shawna Grosskopf, "Theory and Application of Directional Distance Functions," *Journal of Productivity Analysis*, Vol. 13, No. 2 (March 2000), pp. 93-103.

[3] Färe, Rolf, Shawna Grosskopf, and Osman Zaim, "Hyperbolic Efficiency and Return to the Dollar," *European Journal of Operational Research*, Volume 136 (2002), pp. 671-679.

[4] Färe, Rolf and Daniel Primont, "Luenberger Productivity Indicators: Aggregation Across Firms", forthcoming, *Journal of Productivity Analysis*, 2003.

[5] Luenberger, David G., "New Optimality Principles for Economic Efficiency and Equilibrium," *Journal of Optimization Theory and Applications*, Vol. 75, No. 2 (November 1992), pp. 221-264.

[6] Luenberger, David G., Microeconomic Theory, New York: McGraw-Hill, 1995.